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A GLIMPSE ON THE HYPERBOLA

$y^2 = 35x^2 + 1$

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Abstract:

This paper deals with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous binary quadratic equations with two unknowns $y^2 = 35x^2 + 1$. A few interesting properties among the solutions are given. The construction of second order Ramanujan Numbers is illustrated. Employing the linear combination among the solutions of the given equation, integer solutions for other choices of hyperbola & parabola and a few relations among special polygonal numbers are obtained.

Keywords: Binary quadratic, non-homogeneous quadratic, Pell equation, Positive Pell equation, hyperbola

2010 Mathematics Subject Classification: 11D09

INTRODUCTION:

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-13]. In this communication, yet another interesting hyperbola given by $y^2 = 35x^2 + 1$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

METHOD OF ANALYSIS

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 35x^2 + 1 \tag{1}$$

Whose smallest positive integer solutions is

$$x_0 = 1 \; ; y_0 = 6$$

Whose general solution given by

$$\tilde{x}_n = \frac{1}{2\sqrt{35}} g_n ; \tilde{y}_n = \frac{1}{2} f_n$$

Where

$$\mathrm{f_n} = \left(6 + \sqrt{35}\right)^{\mathrm{n+1}} + \left(6 - \sqrt{35}\right)^{\mathrm{n+1}}$$

$$\mathrm{g_n} = (6 + \sqrt{35})^{\mathrm{n+1}} - (6 - \sqrt{35})^{\mathrm{n+1}}$$

The Recurrence relation satisfied by *x* and *y* are given by

$$x_{n+2} - 12x_{n+1} + x_n = 0 ,$$

$$y_{n+2} - 12y_{n+1} + y_n = 0,$$ where n = 0, 1, 2,

A few numerical examples are given in the following table.1

Table: 1 Numerical values

n	x_n	y_n
0	1	6

1	12	71
2	143	846
3	1704	10081
4	20305	120126
5	241956	1431431

From the above table we observe some interesting properties among the solutions which are presented below:

1. Relations between solutions

- $x_{n+3} - 12x_{n+2} + x_{n+1} = 0$
- $y_{n+1} - x_{n+2} + 6x_{n+1} = 0$
- $y_{n+2} - 6x_{n+2} + x_{n+1} = 0$
- $y_{n+3} - 71x_{n+2} + 6x_{n+1} = 0$
- $2x_{n+2} - \frac{1}{6}[x_{n+1} + x_{n+3}] = 0$
- $2y_{n+1} - \frac{1}{6}[x_{n+3} - 71x_{n+1}] = 0$
- $2y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $2y_{n+3} - \frac{1}{6}[71x_{n+3} - x_{n+1}] = 0$
- $x_{n+2} - 6x_{n+1} - y_{n+1} = 0$
- $x_{n+3} - 71x_{n+1} - 12y_{n+1} = 0$
- $y_{n+2} - 6y_{n+1} - 35x_{n+1} = 0$
- $y_{n+3} - 71y_{n+1} - 420x_{n+1} = 0$
- $x_{n+2} - \frac{1}{3}[x_{n+1} + y_{n+2}] = 0$
- $x_{n+3} - x_{n+1} - 2y_{n+2} = 0$
- $2y_{n+1} - \frac{1}{3}[y_{n+2} - 35x_{n+1}] = 0$
- $2y_{n+3} - \frac{1}{3}[71y_{n+2} + 35x_{n+1}] = 0$

- $2x_{n+2} - \frac{1}{71}[12x_{n+1} + 2y_{n+3}] = 0$
- $x_{n+3} - \frac{1}{71}[x_{n+1} + 12y_{n+3}] = 0$
- $y_{n+1} - \frac{1}{71}[y_{n+3} - 420x_{n+1}] = 0$
- $y_{n+2} - \frac{1}{71}[y_{n+3} - 35x_{n+1}] = 0$

2. Each of the following expressions represents a Perfect square integers:

- $2y_{2n+1} + 2$
- $[24y_{2n+2} - 2y_{2n+3}] + 2$
- $\frac{1}{6}[2y_{2n+2} - 70x_{2n+1}] + 2$
- $[6y_{2n+2} - 70x_{2n+2}] + 2$
- $\frac{1}{11}[142y_{2n+2} - 70x_{2n+3}] + 2$
- $\frac{1}{71}[2y_{2n+3} - 840x_{2n+1}] + 2$
- $[2y_{2n+3} - 140x_{2n+2}] + 2$
- $[142y_{2n+3} - 840x_{2n+3}] + 2$
- $[12x_{2n+1} + 2x_{2n+2}] + 2$
- $\frac{1}{12}[2x_{2n+3} - 142x_{2n+1}] + 2$
- $[12x_{2n+1} - 142x_{2n+2}] + 2$

3. Each of the following expressions represents a cubical integers

- $2y_{3n+2} + 6y_n$
- $24y_{3n+3} - 2y_{3n+4} + 72y_{n+1} - 6y_{n+2}$

- $\frac{1}{6}[2y_{3n+3} - 70x_{3n+2} + 6y_{n+1} - 210x_n]$
- $6y_{3n+3} - 70x_{3n+3} + 18y_{n+1} - 21x_{n+1}$
- $\frac{1}{6}[142y_{3n+3} - 70x_{3n+4} - 426y_{n+1} + 210x_{n+2}]$
- $\frac{1}{71}[2y_{3n+4} - 840x_{3n+2} + 6y_{n+2} - 2520x_n]$
- $2y_{3n+4} - 140x_{3n+3} + 6y_{n+2} - 420x_{n+1}$
- $142y_{3n+4} - 840x_{3n+4} + 426y_{n+2} - 2520x_{n+2}$
- $12x_{3n+2} - 2x_{3n+3} + 36x_n - 6x_{n+1}$
- $\frac{1}{12}[2x_{3n+2} - 142x_{3n+3} + 6x_{n+2} - 426x_n]$
- $12x_{3n+2} - 142x_{3n+3} + 36x_{n+2} - 426x_{n+1}$

4. Each of the following expressions represents a Bi-quadratic Integer

- $2y_{4n+3} + 16y_n^2 - 2$
- $24[y_{n+4} - 2y_{4n+5}] + [96y_{2n+2} - 8y_{2n+3}] + 6$
- $\frac{1}{6}[[2y_{4n+4} - 70x_{4n+3}] + [8y_{2n+2} - 280x_{2n+1}] + 6]$
- $[6y_{4n+4} - 70x_{4n+4}] + [24y_{2n+2} - 28x_{2n+2}] + 6$
- $\frac{1}{6}[142y_{4n+4} - 70x_{4n+5}] - [568y_{2n+2} + 28x_{2n+3}] + 6]$
- $\frac{1}{71}[[2y_{4n+5} - 840x_{4n+3}] + [8y_{2n+3} - 3360x_{2n+3}] + 6]$
- $2[y_{4n+5} - 140x_{4n+4}] + [8y_{2n+3} - 560x_{2n+2}] + 6$
- $[142y_{4n+5} - 840x_{4n+5}] + [568y_{2n+3} - 3360x_{2n+3}] + 6$
- $[12x_{4n+3} - 2x_{4n+4}] + [48x_{2n+1} - 8x_{2n+2}] + 6$
- $\frac{1}{12}[[2x_{4n+3} - 142x_{4n+4}] + [8x_{2n+1} - 568x_{2n+2}] + 6]$
- $[12x_{4n+3} - 142x_{4n+4}] + [48x_{2n+1} - 568x_{2n+2}] + 6$

5. Each of the following expressions represents a Quintic Integer:

- $[2y_{5n+4} + 5(2y_n)^3 - 5(2y_n)$
- $[24y_{5n+5} - 2y_{5n+6}] + 5[24y_{n+1} - 2y_{n+2}\ ]^3 - 5[24y_{n+1} - 2y_{n+2}]$
- $\frac{1}{6}[2y_{5n+4} - 70x_{5n+4}] + 5[\frac{1}{6}[2y_{n+1} - 70x_n\ ]^3 - 5[\frac{1}{6}(2y_{n+1} - 70x_n)]$
- $[6y_{5n+5} - 70x_{5n+5}] + 5[6y_{n+1} - 70x_{n+1}\ ]^3 - 5[6y_{n+1} - 70x_{n+1}]$
- $\frac{1}{6}[142y_{5n+5} - 70x_{5n+6}] + 5[\frac{1}{6}[142y_{n+1} - 70x_{n+2}\ ]^3 - 5[\frac{1}{6}(142y_{n+1} - 70x_{n+2})]$
- $\frac{1}{71}[2y_{5n+6} - 840x_{5n+4}] + 5[\frac{1}{71}[(2y_{n+2} - 840x_n)]^3 - 5[\frac{1}{71}(2y_{n+2} - 840x_n)]$
- $[2y_{5n+6} - 140y_{5n+5}] + 5[2y_{n+2} - 140x_{n+1}]^3 - 5[2y_{n+2} - 140x_{n+1}]$
- $[142y_{5n+6} - 840y_{5n+6}] + 5[142y_{n+2} - 840x_{n+2}]^3 - 5[142y_{n+2} - 840x_{n+2}]$
- $[12x_{5n+4} - 2x_{5n+5}] + 5[(12x_n - 2x_{n+1})]^3 - 5[(12x_n - 2x_{n+1})]$
- $\frac{1}{6}[x_{5n+6} - 71x_{5n+4}] + 5[\frac{1}{6}[(x_{n+2} - 71x_n)]^3 - 5[\frac{1}{6}(x_{n+2} - 71x_n)]$
- $[12x_{5n+6} - 142x_{5n+5}] + 5[(12x_{n+2} - 142x_{n+1})]^3 - 5[(12x_{n+2} - 142x_{n+1})]$

6. Construction of second order Ramanujan Number:

The process of obtaining second order Ramanujan Number from suitable choices of x and y is illustrated through an example below.

From

$x_1 = 12$

= 12*1 = 4*3 = 6*2

= A = B = C

From A = B consider the relation

$$(12+1)^2 + (4-3)^2 = (12-1)^2 + (4+3)^2$$

$$(13)^2 + (1)^2 = (11)^2 + (7)^2 = 170$$

From A = C consider the relation

$$(12+1)^2 + (6-2)^2 = (12-1)^2 + (6+2)^2$$

$$(13)^2 + (4)^2 = (11)^2 + (8)^2 = 185$$

From B = C consider the relation

$$(4+3)^2 + (6-2)^2 = (4-3)^2 + (6+2)^2$$

$$(7)^2 + (4)^2 = (1)^2 + (8)^2 = 65$$

Each of the numbers 170,185,65 is expressed as sum of two squares in two different ways.

Thus the above number are second order Ramanujan Numbers.

REMARKABLE OBSERVATIONS:

1. Employing linear combinations among the solutions of (1), one may generate solution for other choices of hyperbola which are presented in table :2 below

Table: 2 Hyperbola

S.No	Hyperbola	(X,Y)
1.	$35X^2 - Y^2 = 35$	$X = y_n$ $Y = y_{n+1} - 6y_n$
2.	$5040X^2 - Y^2 = 5040$	$X = y_n$ $Y = y_{n+2} - 71y_n$
3.	$X^2 - 35Y^2 = 1$	$X = y_n$ $Y = x_n$
4.	$36X^2 - 35Y^2 = 36$	$X = y_n$ $Y = x_{n+1} - y_n$
5.	$5041X^2 - 35Y^2 = 5041$	$X = y_n$ $Y = x_{n+2} - 12y_n$
6.	$35X^2 - Y^2 = 35$	$X = (6y_{n+1} - y_{n+2})$ $Y = (6y_{n+2} - 71y_{n+2})$
7.	$36X^2 - 5040Y^2 = 144$	$X = (y_{n+1} - 35x_n)$ $Y = x_n$
8.	$X^2 - 35Y^2 = 1$	$X = (3y_{n+1} - 35x_{n+1})$ $Y = (6x_{n+1} - y_{n+1})$

9.	$X^2 - 35Y^2 = 36$	$X = (71y_{n+1} - 35x_{n+2})$ $Y = (36x_{n+2} - 72y_{n+1})$
10.	$X^2 - 176435Y^2 = 5041$	$X = (y_{n+2} - 420x_n)$ $Y = x_n$
11.	$X^2 - 35\,Y^2 = 36$	$X = (36y_{n+2} - 2520x_{n+1})$ $Y = (71x_{n+1} - y_{n+2})$
12.	$X^2 - Y^2 = 4$	$X = (142y_{n+2} - 840x_{n+2})$ $Y = (142x_{n+2} - 840y_{n+2})$
13.	$35X^2 - 1225Y^2 = 35$	$X = (6x_n - x_{n+1})$ $Y = x_n$
14.	$X^2 - 210Y^2 = 6$	$X = (x_{n+2} - 71x_n)$ $Y = x_n$
15.	$X^2 - 35Y^2 = 1$	$X = (6x_{n+2} - 71x_{n+1})$ $Y = (12x_{n+1} - x_{n+2})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table:3 below:

Table:3 Parabola

S.No	Parabola	(X,Z)
1.	$70X - 4Z^2 = 140$	$X = (y_{2n+1} + 1)$ $Z = (y_{n+1} - 6y_n)$
2.	$2520X - Z^2 = 5040$	$X = (y_{2n+1} + 1)$ $Z = (y_{n+2} - 71y_n)$
3.	$X - 70Z^2 = 2$	$X = (y_{2n+1} + 1)$ $Z = 70x_n$

4.	$18X - 35Z^2 = 36$	$X = (y_{2n+1} + 1)$ $Z = (x_{n+1} - y_n)$
5.	$5041X - 70Z^2 = 5041$	$X = (y_{2n+1} + 1)$ $Z = (2x_{n+2} - 12y_n)$
6.	$35X - 2Z^2 = 140$	$X = (12y_{2n+2} - y_{2n+3} + 1)$ $Z = (6y_{n+2} - 71y_{n+1})$
7.	$X - 840Z^2 = 12$	$X = (2y_{2n+2} - 70x_{2n+1})$ $Z = x_n$
8.	$X - 70Z^2 = 2$	$X = (3y_{2n+2} - 35x_{2n+2} + 1)$ $Z = (6x_{n+1} - y_{n+1})$
9.	$X - 70Z^2 = 12$	$X = (71y_{2n+2} - 35x_{2n+3} + 1)$ $Z = (6x_{n+2} - 12y_{n+1})$
10.	$X - 9940Z^2 = 142$	$X = (y_{2n+3} - 420x_{2n+1} + 1)$ $Z = x_n$
11.	$X - 35Z^2 = 12$	$X = (6y_{2n+3} - 420x_{2n+2} + 1)$ $Z = (142x_{n+1} - 2y_{n+2})$
12.	$X - 2Z^2 = 2$	$X = (71y_{2n+3} - 420x_{2n+3})$ $Z = (71x_{n+2} - 420y_{n+2})$
13.	$X - 70Z^2 = 2$	$X = (6x_{2n+1} - x_{2n+2} + 1)$ $Z = x_n$
14.	$X - 105Z^2 = 3$	$X = (x_{2n+1} - 71x_{2n+2} + 1)$ $Z = x_n$
15.	$X - 70\,Z^2 = 2$	$X = (6x_{2n+1} - 71x_{2n+2} + 1)$ $Z = (12x_{n+1} - x_{n+2})$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 35x^2 + 1$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive pell equations and determine the integer solutions along with suitable properties.

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